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Kontsevich Deformation Quantization and Flat Connections

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Abstract: In Torossian (J Lie Theory 12(2):597–616, 2002), the second author used the Kontsevich deformation quantization technique to define a natural connection ω_n on the compactified configuration spaces $\overline{C}_{n,0}$ of n points on the upper half-plane. Connections ω_n take values in the Lie algebra of derivations of the free Lie algebra with n generators. In this paper, we show that ω_n is flat.

The configuration space $\overline{C}_{n,0}$ contains a boundary stratum at infinity which coincides with the (compactified) configuration space of n points on the complex plane. When restricted to this stratum, ω_n gives rise to a flat connection ω_n^∞. We show that the parallel transport Φ defined by the connection ω_3^∞ between configuration $1(23)$ and $(12)3$ verifies axioms of an associator.

We conjecture that ω_n^∞ takes values in the Lie algebra $\mathfrak{t}_n$ of infinitesimal braids. If correct, this conjecture implies that $\Phi \in \exp(\mathfrak{t}_3)$ is a Drinfeld's associator. Furthermore, we prove $\Phi \neq \Phi_{KZ}$ showing that Φ is a new explicit solution of associator axioms.

1. Introduction

The Kontsevich proof of the formality conjecture and the construction of the star product on $\mathbb{R}^d$ equipped with a given Poisson structure make use of integrals of certain differential forms over compactified configuration spaces $\overline{C}_{n,m}$ of points on the upper half-plane. Here n points are free to move in the upper half-plane, m points are bound to the real axis, and we quotient by the diagonal action of the group $z \mapsto az + b$ with $a \in \mathbb{R}_+, b \in \mathbb{R}$.

In this paper, we use the same ingredients to study a certain connection ω_n on $\overline{C}_{n,0}$ with values in the Lie algebra of derivations of the free Lie algebra with n generators. This connection was introduced by the second author in [14]. One of our results is flatness of ω_n.

The compactified configuration space $\overline{C}_{n,0}$ contains a boundary stratum "at infinity" which coincides with the configuration space of n points on the complex plane (quotient by the diagonal action $z \mapsto az + b$ with $a \in \mathbb{R}_+, b \in \mathbb{C}$). Over this boundary stratum, the connection ω_n restricts to the connection ω_n^∞ with values in the Lie algebra $\mathfrak{krv}_n$ defined in [3]. We conjecture that in fact ω_n^∞ takes values in the Lie algebra $\mathfrak{t}_n \subset \mathfrak{krv}_n$ defined by the infinitesimal braid relations.

Let Φ be the parallel transport defined by the connection ω_3^∞ for the straight path between configurations 1(23) and (12)3 of 3 points on the complex plane. We show that Φ verifies axioms of an associator with values in the group $KRV_3 = \exp(\mathfrak{krv}_3)$. If the conjecture of the previous paragraph holds true, then $\Phi \in \exp(\mathfrak{t}_3)$, and it becomes a Drinfeld's associator. The key ingredient in the proof of the pentagon axiom is the flatness property of ω_n^∞. The construction of Φ is parallel to the construction of the Knizhnik-Zamolodchikov associator Φ_{KZ} in [7] with ω_3^∞ replacing the Knizhnik-Zamolodchikov connection. Furthermore, one can show that Φ is even, and hence $\Phi \neq \Phi_{KZ}$.

While this paper was in preparation, we learnt of the work [12] proving our conjecture stated above.

The plan of the paper is as follows. In Sect. 2, we review some standard facts about the Kontsevich déformation quantization technique and free Lie algebras. In Sect. 3, we prove flatness of the connection ω_n. Section 4 contains the proof of associator axioms for the element Φ.

2. Deformation Quantization and Free Lie Algebras

Many sources are now available on the Kontsevich formula for quantization of Poisson brackets (see *e.g.* [5]). For the convenience of the reader, we briefly recall the main ingredients of [11] for $\mathbb{R}^d$ and the construction [14] of the connection ω_n.

2.1. Free Lie algebras and their derivations.

2.1.1. Free Lie algebras and derivations. Let $\mathbb{K}$ be a field of characteristic zero, and let $\mathfrak{lie}_n = \mathfrak{lie}(x_1, \ldots, x_n)$ be the degree completion of the graded free Lie algebra over $\mathbb{K}$ with generators $x_1, \ldots, x_n$ of degree one. We shall denote by $\mathfrak{der}_n$ the Lie algebra of derivations of $\mathfrak{lie}_n$. An element $u \in \mathfrak{der}_n$ is completely determined by its values on generators, $u(x_1), \ldots, u(x_n) \in \mathfrak{lie}_n$. The Lie algebra $\mathfrak{der}_n$ carries a grading induced by the one of $\mathfrak{lie}_n$.

Definition 1. *A derivation $u \in \mathfrak{der}_n$ is called* **tangential** *if there exist $a_i \in \mathfrak{lie}_n, i = 1, \ldots, n$ such that $u(x_i) = [x_i, a_i]$.*

Tangential derivations form a Lie subalgebra $\mathfrak{tder}_n \subset \mathfrak{der}_n$. Elements of $\mathfrak{tder}_n$ are in one-to-one correspondence with n-tuples of elements of $\mathfrak{lie}_n$, $(a_1, \ldots, a_n)$, which verify the condition that a_k has no linear term in x_k for all k. By abuse of notations, we shall often write $u = (a_1, \ldots, a_n)$. For two elements of $\mathfrak{tder}_n$, $u = (a_1, \ldots, a_n)$ and $v = (b_1, \ldots, b_n)$, we have $[u, v]_{\mathfrak{tder}} = (c_1, \ldots, c_n)$ with

$$c_k = u(b_k) - v(a_k) + [a_k, b_k]_{\mathfrak{lie}}. \tag{1}$$

Definition 2. *A derivation $u = (a_1, \ldots, a_n) \in \mathfrak{tder}_n$ is called* **special** *if $u(x) = \sum_i [x_i, a_i] = 0$ for $x = \sum_{i=1}^n x_i$.*

We shall denote the space of special derivations by $\mathfrak{sder}_n$. It is obvious that $\mathfrak{sder}_n \subset \mathfrak{tder}_n$ is a Lie subalgebra. Both $\mathfrak{tder}_n$ and $\mathfrak{sder}_n$ integrate to prouniportent groups denoted by $TAut_n$ and $SAut_n$, respectively. In more detail, $TAut_n$ consists of automorphisms of $\mathfrak{lie}_n$ such that $x_i \mapsto \mathrm{Ad}_{g_i} x_i = g_i x_i g_i^{-1}$, where $g_i \in \exp(\mathfrak{lie}_n)$. Similarly, elements of $SAut_n$ are tangential automorphisms of $\mathfrak{lie}_n$ with an extra property $x = \sum_{i=1}^n x_i \mapsto x$.

The family of Lie algebras $\mathfrak{tder}_n$ is equipped with simplicial Lie homomorphisms $\mathfrak{tder}_n \to \mathfrak{tder}_{n+1}$. For instance, for $u = (a, b) \in \mathfrak{tder}_2$ we define

$$\begin{aligned} u^{1,2} &= (a(x, y), b(x, y), 0), \\ u^{2,3} &= (0, a(y, z), b(y, z)), \\ u^{12,3} &= (a(x + y, z), a(x + y, z), b(x + y, z)), \end{aligned}$$

and similarly for other simplicial maps. These Lie homomorphisms integrate to group homomorphisms of $TAut_n$ and $SAut_n$.

2.1.2. Cyclic words. Let $Ass_n^+ = \prod_{k=1}^\infty Ass^k(x_1, \ldots, x_n)$ be the graded free associative algebra (without unit) with generators $x_1, \ldots, x_n$. Every element $a \in Ass_n^+$ admits a unique decomposition of the form $a = \sum_{i=1}^n (\partial_i a) x_i$, where $\partial_i a \in Ass_n$ (Ass_n is a free associative algebra with unit).

We define the graded vector space cy_n as a quotient

$$\mathrm{cy}_n = Ass_n^+ / \langle (ab - ba); a, b \in Ass_n \rangle.$$

Here $\langle (ab - ba); a, b \in Ass_n \rangle$ is the subspace of Ass_n^+ spanned by commutators. The multiplication map of Ass_n^+ does not descend to cy_n which only has a structure of a graded vector space. We shall denote by $\mathrm{tr} : Ass_n^+ \to \mathrm{cy}_n$ the natural projection. By definition, we have $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in Ass_n$ imitating the defining property of trace. In general, graded components of cy_n are spanned by words of a given length modulo cyclic permutations.

Example 1. *The space* cy_1 *is isomorphic to the space of formal power series in one variable without constant term,* $\mathrm{cy}_1 \cong xk[[x]]$. *This isomorphism is given by the following formula:*

$$f(x) = \sum_{k=1}^\infty f_k x^k \mapsto \sum_{k=1}^\infty f_k \mathrm{tr}(x^k).$$

2.1.3. Divergence. Let $u = (a_1, \ldots, a_n) \in \mathfrak{tder}_n$. We define the divergence as

$$\mathrm{div}(u) = \sum_{i=1}^n \mathrm{tr}(x_i (\partial_i a_i)).$$

It is a 1-cocycle of $\mathfrak{tder}_n$ with values in cy_n (see Proposition 3.6 in [3]).

We define $\mathfrak{krv}_n \subset \mathfrak{sder}_n \subset \mathfrak{tder}_n$ as the Lie algebra of special derivation with vanishing divergence. Hence, $u = (a_1, \ldots, a_n) \in \mathfrak{krv}_n$ is a solution of two equations: $\sum_{i=1}^n [x_i, a_i] = 0$ and $\sum_{i=1}^n \mathrm{tr}(x_i(\partial_i a_i)) = 0$. We shall denote by $KRV_n = \exp(\mathfrak{krv}_n)$ the corresponding prounipotent group.

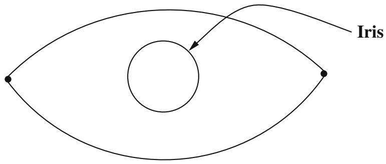

Fig. 1. Variety $\overline{C}_{2,0}$

2.2. Kontsevich construction.

2.2.1. Configurations spaces. We denote by $C_{n,m}$ the configuration space of n distinct points in the upper half plane and m points on the real line modulo the diagonal action of the group $z \mapsto az + b$ ($a \in \mathbb{R}_+$, $b \in \mathbb{R}$). In [11], Kontsevich constructed compactifications of spaces $C_{n,m}$ denoted by $\overline{C}_{n,m}$. These are manifolds with corners of dimension $2n - 2 + m$. We denote by $\overline{C}^+_{n,m}$ the connected component of $\overline{C}_{n,m}$ with real points in the standard order (*id.* $\overline{1} < \overline{2} < \cdots < \overline{m}$).

The compactified configuration space $\overline{C}_{2,0}$ (the "Kontsevich eye") is shown on Fig. 1. The upper and lower eyelids correspond to one of the points (z_1 or z_2) on the real line, left and right corners of the eye are configurations with $z_1, z_2 \in \mathbb{R}$ and $z_1 > z_2$ or $z_1 < z_2$. The boundary of the iris takes into account configurations where z_1 and z_2 collapse inside the complex plane. The angle along the iris keeps track of the angle at which z_1 approaches z_2.

2.2.2. Graphs. The Kontsevich graphical calculus (in the case of linear Poisson brackets) was studied in [4] and [10]. A graph Γ is a collection of vertices V_Γ and oriented edges E_Γ. Vertices are ordered, and the edges are ordered in a way compatible with the order of the vertices. We denote by $G_{n,2}$ the set of graphs with $n + 2$ vertices and $2n$ edges verifying the following properties:

- i - There are n vertices of the first type $1, 2, \ldots, n$ and 2 vertices of the second type $\overline{1}, \overline{2}$.
- ii - Edges start from vertices of the first type, 2 edges per vertex.
- iii - Source and target of an edge are distinct.
- iv - There are no multiple edges (same source and target).

We are interested in the case of *linear graphs*. That is, vertices of the first type admit at most one incoming edge. Such graphs are superpositions of simple graphs of two types, Lie type graphs (graphs with one root as on Fig. 2) and wheel type graphs (graph with one oriented loop, as on Fig. 3).

2.2.3. The angle map and Kontsevich weights. Let p and q be two points on the upper half plane. Consider the hyperbolic angle map on $C_{2,0}$:

$$\phi_h(p, q) = \arg\left(\frac{q - p}{q - \overline{p}}\right) \in \mathbb{T}^1. \tag{2}$$

This function admits a continuous extension to the compactification $\overline{C}_{2,0}$.

Consider a graph $\Gamma \in G_{n,2}$, and draw it in the upper half plane with vertices of the second type on the real line. By restriction, each edge e defines an angle map ϕ_e on $\overline{C}^+_{n,2}$.

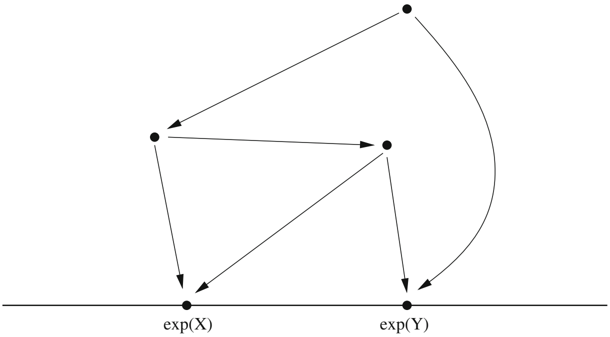

Fig. 2. Lie type graph with symbol $\Gamma(x, y) = [[x, [x, y]], y]$

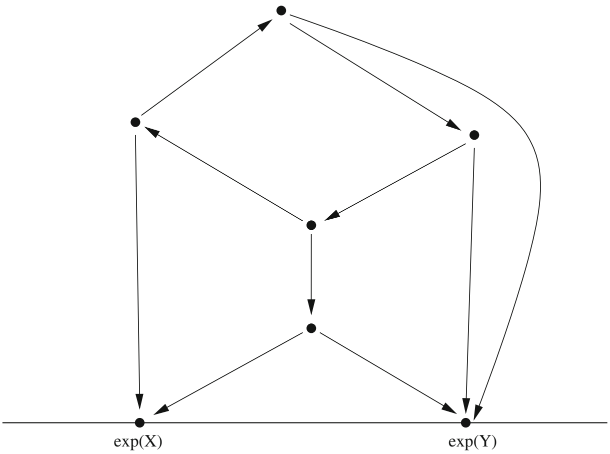

Fig. 3. Wheel type graph with symbol $\Gamma(x, y) = \mathrm{tr}(y^2[x, y]x)$

The ordered product

$$\Omega_\Gamma = \bigwedge_{e \in E_\Gamma} \mathrm{d}\phi_e \tag{3}$$

is a regular $2n$-form on $\overline{C}^+_{n,2}$ (which is a $2n$-dim compact space).

Definition 3. *The Kontsevich weight of* Γ *is given by the following formula*:

$$w_\Gamma = \frac{1}{(2\pi)^{2n}} \int_{\overline{C}^+_{n,2}} \Omega_\Gamma. \tag{4}$$

2.3. Campbell-Hausdorff and Duflo formulas. Lie type graphs in $G_{n,2}$ are binary rooted trees. Hence, to each $\Gamma \in G_{n,2}$ a simple Lie type graph one can associate a Lie word $\Gamma(x, y) \in \mathfrak{lie}_2$ of degree $2n$ in variables x, y (see Fig. 2). Similarly, if Γ is a wheel type graph, it corresponds to an element $\Gamma(x, y) \in \mathrm{cy}_2$ (see Fig. 3).

Recall the definition of the Duflo density function

$$\mathrm{duf}(x,y) = \frac{1}{2}\left(j(x) + j(y) - j(\mathrm{ch}(x,y))\right) \in \mathrm{cy}_2,$$

where $\mathrm{ch}(x,y) = \log(e^x e^y)$ is the Campbell-Hausdorff series and

$$j(x) = \sum_{n\geq 2} \frac{b_n}{n\cdot n!}\mathrm{tr}(x^n)$$

with b_n the Bernoulli numbers. The following theorem relates functions $\mathrm{ch}(x,y)$ and $\mathrm{duf}(x,y)$ to the Kontsevich graphical calculus.

Theorem 1 ([10], [4]). *The following identities hold true*:

$$\mathrm{ch}(x,y) = x + y + \sum_{n\geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{Lie type } (n,2)}} w_\Gamma \Gamma(x,y), \tag{5}$$

$$\mathrm{duf}(x,y) = \sum_{n\geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{wheel type } (n,2)}} \frac{w_\Gamma}{m_\Gamma}\Gamma(x,y), \tag{6}$$

where m_Γ is the order of the symmetry group of the graph Γ.

Here *geometric* means that graphs are not labeled. Note that the definition of both $\Gamma(x,y)$ and w_Γ requires an order on the set of edges, but the product $w_\Gamma\Gamma(x,y)$ is independent of this order. Even though $\mathrm{ch}(x,y)$ and $\mathrm{duf}(x,y)$ are defined over rationals, some of the coefficients w_Γ are very probably irrational (see example of [8]).

Remark 1. Note that the associativity of the Kontsevich star product implies that the right hand side of Eq. (5) is an associative Lie series. If we denote it by $\chi(x,y)$, we have $\chi(\chi(x,y),z) = \chi(x,\chi(y,z))$. Then, $\chi(x,y)$ coincides (up to rescaling of arguments) with the Campbell-Hausdorff series (see *e.g.* Proposition 2.1 in [3]).

Remark 2. Denote the right hand side of Eq. (6) by $\Delta(x,y)$. Similarly to the previous remark, the associativity of the star product implies

$$\Delta(x,y) + \Delta(\mathrm{ch}(x,y),z) = \Delta(x,\mathrm{ch}(y,z)) + \Delta(y,z).$$

By Proposition 2.2 in [3], this gives $\Delta(x,y) = f(x) + f(y) - f(\mathrm{ch}(x,y))$. Finally, by looking at degree one in y contributions (see Remark 8.5.5 in [5], and also [6]) one arrives at $f(x) = \sum_n w_n \mathrm{tr}(x^n)/n$ with coefficients w_n given by Kontsevich graphs presented on Fig. 4. To the best of our knowledge, there is no direct computation of these graphs available in literature.

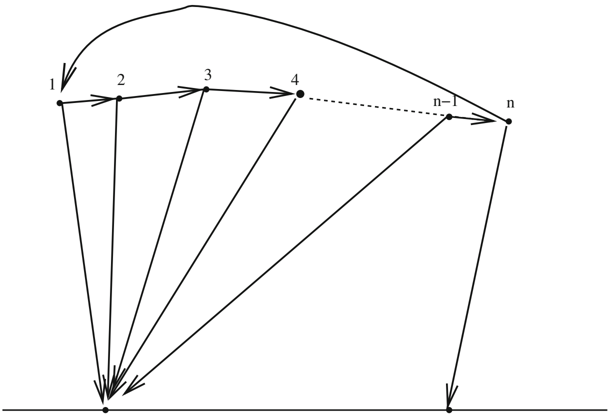

Fig. 4. Coefficient w_n

2.4. ξ-deformation. In [14], one studies the following deformation for the Campbell-Hausdorff formula. Let $\xi \in \overline{C}_{2,0}$, $\Gamma \in G_{n,2}$, and let π be the natural projection from $\overline{C}_{n+2,0}$ onto $\overline{C}_{2,0}$. We define the coefficients $w_\Gamma(\xi)$ for $\xi \in \overline{C}_{2,0}$ as

$$w_\Gamma(\xi) = \frac{1}{(2\pi)^{2n}} \int_{\pi^{-1}(\xi)} \Omega_\Gamma.$$

Functions $w_\Gamma(\xi)$ are smooth over $C_{2,0}$, and they are continuous over the compactification $\overline{C}_{2,0}$. The ξ-deformation of the Campbell-Hausdorff series $\mathrm{ch}_\xi(x, y)$ is defined as

$$\mathrm{ch}_\xi(x, y) = x + y + \sum_{n\geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{Lie type } (n,2)}} w_\Gamma(\xi)\Gamma(x, y). \tag{7}$$

In a similar fashion, we introduce a deformation of the Duflo function,

$$\mathrm{duf}_\xi(x, y) = \sum_{n\geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{wheel type } (n,2)}} \frac{w_\Gamma(\xi)}{m_\Gamma}\Gamma(x, y).$$

For $\xi = (0, 1)$ (the right corner of the eye on Fig. 1), the expression (7) is given by the standard Campbell-Hausdorff series, and for ξ in the position α on the iris, the Kontsevich Vanishing Lemma implies $\mathrm{ch}_\alpha(x, y) = x+y$. By the results of [4] and [13], for $\xi = (0, 1)$ the Duflo function $\mathrm{duf}_\xi(x, y)$ coincides with the standard Duflo function, and for ξ in the arbitrary position α on the iris one has $\mathrm{duf}_\alpha(x, y) = 0$.

2.5. Connection ω_2. In [14], one defines a connection on $C_{2,0}$ with values in $\mathfrak{tder}_2$,

$$\omega_2 = \left(F_\xi(x, y), G_\xi(x, y)\right).$$

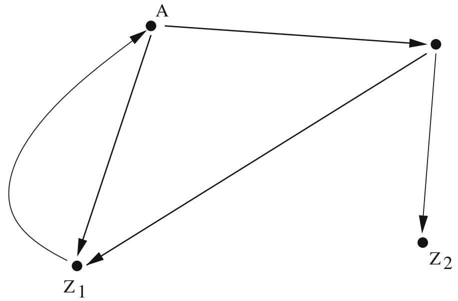

Fig. 5. Extended graph

Here F_ξ and G_ξ are 1-forms on $C_{2,0}$ taking values in $\mathfrak{lie}_2$. They satisfy the following two (Kashiwara-Vergne type) equations (see Theorem 1 and Theorem 2 in [14])

$$\mathrm{d}\,\mathrm{ch}_\xi(x,y) = \omega_2(\mathrm{ch}_\xi(x,y)), \tag{8}$$

$$\mathrm{d}\,\mathrm{duf}_\xi(x,y) = \omega_2(\mathrm{duf}_\xi(x,y)) + \mathrm{div}(\omega_2), \tag{9}$$

where d is the de Rham differential on $C_{2,0}$, and ω_2 acts on $\mathrm{ch}_\xi(x,y)$ and $\mathrm{duf}_\xi(x,y)$ as a derivation of $\mathfrak{lie}_2$.

Let us briefly recall the construction of ω_2. We will denote by $A, B \in G_{n,2}$ simple graphs of Lie type, and we define an extended graph $\ulcorner\!\!\to\! A$ (resp. $B \leftarrow\!\!\urcorner$) as a graph with an additional edge starting at $\overline{1}$ (resp. $\overline{2}$) and ending at the root of A (resp. B), see Fig. 5. Note that $n = 0$ is allowed, but since the source and the target of an edge must be distinct, Γ will have a single edge starting at $\overline{1}$ and ending at $\overline{2}$, or starting at $\overline{2}$ and ending at $\overline{1}$.

Draw the extended graph in the upper half plane (with vertices of the second type corresponding to $\xi \in \overline{C}_{2,0}$). Then,

$$\Omega_{\ulcorner\!\!\to A} = \bigwedge_{e \in E_{\ulcorner\!\!\to A}} \mathrm{d}\phi_e$$

is a $2n+1$-form on $\overline{C}_{n+2,0}$ (which is a $2n+2$-dim compact space). The push forward along the natural projection $\pi : \overline{C}_{n+2,0} \to \overline{C}_{2,0}$ yields a 1-form on $\overline{C}_{2,0}$, $\omega_{\ulcorner\!\!\to A} = \pi_*(\Omega_{\ulcorner\!\!\to A})$.

The connection $\omega_2 = (F_\xi(x,y), G_\xi(x,y))$ is defined by the following formula,[1]

$$\begin{cases} F_\xi(x,y) = \sum\limits_{n\geq 0} \sum\limits_{\substack{A \text{ simple} \\ \text{graph of} \\ \text{Lie type } (n,2)}} \omega_{\ulcorner\!\!\to A} A(x,y) \\ G_\xi(x,y) = \sum\limits_{n\geq 0} \sum\limits_{\substack{B \text{ simple} \\ \text{graph of} \\ \text{Lie type } (n,2)}} \omega_{B\leftarrow\!\!\urcorner} B(x,y) \end{cases} \tag{10}$$

2.6. Definition of ω_n and ω_n^∞. We now extend this construction to an arbitrary number of vertices of the second type.

Consider $\Gamma \in G_{p,n}$ a simple graph of Lie type with vertices of the second type labeled $\overline{1}, \ldots, \overline{n}$. Define the extended graph $\Gamma^{(i)}$ by adding an edge from the vertex $\overline{i}$ to the root

[1] For $n = 0$, $A = y$ and $B = x$.

of Γ. Consider the natural projection $\pi : \overline{C_{p+n,0}} \to \overline{C_{n,0}}$ and take the pushforward 1-form $\omega_{\Gamma^{(i)}} = \pi_*(\Omega_{\Gamma^{(i)}})$. We define 1-forms with values in $\mathfrak{lie}_n$,

$$F_i = \sum_{j\neq i} \frac{\mathrm{d}\phi_h(z_i, z_j)}{2\pi} x_j + \sum_{p>0} \sum_{\substack{\Gamma \text{ simple} \\ \text{graph of} \\ \text{Lie type } (p,n)}} \omega_{\Gamma^{(i)}} \Gamma(x_1, \ldots, x_n).$$

Here the first term gives an explicit expression of the $p = 0$ contribution. The expression $\omega_n = (F_1, \ldots, , F_n)$ defines a connection with values in $\mathfrak{tder}_n$.

The connection ω_n is smooth over $C_{n,0}$. Over the compactification $\overline{C}_{n,0}$, it belongs to the class L^1 when restricted to piece-wise differentiable curves. Hence, along such curves all iterated integrals converge, and there is a unique solution of the initial value problem $\mathrm{d}g = -g\omega$ with $g(z_0) = 1$ for the base point z_0 (e.g. by using Grönwall's Lemma). Therefore, parallel transports are well defined.

The same applies to restrictions of ω_n to boundary strata of $\overline{C}_{n,0}$ of dimension at least one. For instance, in the case of $\overline{C}_{2,0}$ one can consider a path along the eyelid, or a generic path from the corner of $\overline{C}_{2,0}$ to the iris.

We will need restrictions of ω_n to various boundary strata of co-dimension one of $\overline{C}_{n,0}$. First of all, there is a stratum "at infinity" equal to the configuration space C_n of n points on the complex plane (modulo the diagonal action of the group $z \mapsto az + b$ for $a \in \mathbb{R}_+, b \in \mathbb{C}$). We denote the corresponding connection ω_n^∞. It is given by the same formula as ω_n with the configuration space $\overline{C}_{n,0}$ replaced by $\overline{C}_n$, and the hyperbolic angle is replaced with the Euclidean angle. In particular, we have

$$\omega_n^\infty = \sum_{i<j} t_{i,j} \frac{\mathrm{d}\phi_e(z_i, z_j)}{2\pi} + \cdots ,$$

where $t_{i,j} = (0, \ldots, x_j, 0, \ldots, x_i, \ldots, 0)$ with x_j placed at the position i and x_i at the position j.

Next, for the first q points collapsing inside the upper half plane, we have a stratum of the form $C_q \times C_{n-q+1,0}$. We denote the natural projections by π_1 and π_2, and obtain an expression for the connection

$$\omega_n|_{C_q \times C_{n-q+1,0}} = \pi_1^*(\omega_q^\infty)^{1,2,\ldots,q} + \pi_2^* \omega_{n-q+1}^{12\ldots q,q+1,\ldots,n}.$$

Other choices of points to collapse can be described by using the action of the symmetric group S_n.

A similar property holds for the connection ω_n^∞ on the stratum $C_q \times C_{n-q+1}$ corresponding to (the first) q points collapsing together,

$$\omega_n^\infty|_{C_q \times C_{n-q+1}} = \pi_1^*(\omega_q^\infty)^{1,2,\ldots,q} + \pi_2^*(\omega_{n-q+1}^\infty)^{12\ldots q,q+1,\ldots,n}.$$

In the case when (the first) q points are collapsing to a point on the real axis, we obtain the stratum $C_{q,0} \times C_{n-q,1}$, and for the connection we get

$$\omega_n|_{C_{q,0} \times C_{n-q,1}} = \pi_1^* \omega_q^{1,2,\ldots,q} + \pi_2^* \omega_{n-q+1}^{12\ldots q,q+1,\ldots,n}|_{C_{n-q,1}}.$$

Note that the restriction of the connection form ω_n to the boundary stratum $C_{n-1,1}$ corresponds to configurations with the point z_1 on the real axis, and it has the following property: its first component (as an element of $\mathfrak{tder}_n$) vanishes since the 1-form $d\phi_e$ vanishes when the source of the edge e is bound to the real axis.

3. Zero Curvature Equation and Applications

One of our main results is flatness of the connection ω_n.

3.1. The zero curvature equation.

Theorem 2. *The connection ω_n is flat. That is, the following 2-form on $C_{n,0}$ vanishes:*

$$d\omega_n + \frac{1}{2}[\omega_n, \omega_n] = 0. \tag{11}$$

Proof. The argument is based on the Stokes formula, and we give details in the case of ω_2. The case of arbitrary n is treated in a similar fashion.

Let C_ξ be a small circle around $\xi \in C_{2,0}$, Δ_ξ be the corresponding disk, and consider $\pi^{-1}(\Delta_\xi)$. Since the forms $\Omega_{\Rsh A}$, $\Omega_{B\Lsh}$ are closed, we have

$$\int_{\pi^{-1}(\Delta_\xi)} d\left(\sum_A \Omega_{\Rsh A} A(x,y), \sum_B \Omega_{B\Lsh} B(x,y)\right) = 0.$$

By applying Stokes formula and the definition of the connection ω_2, one obtains

$$0 = \int_{C_\xi} \omega_2 + \int_{\bigcup_{z\in\Delta_\xi} \partial(\pi^{-1}(z))} \left(\sum_A \Omega_{\Rsh A} A(x,y), \sum_B \Omega_{B\Lsh} B(x,y)\right). \tag{12}$$

By using again Stokes's formula (on the disk Δ_ξ), one can rewrite the first term in the form $\int_{\Delta_\xi} d_\xi \omega_2$.

For the second term, one obtains contributions from the boundary strata of codimension one. The usual arguments in Kontsevich theory rule out strata where more than two points collapse (by the Kontevich Vanishing Lemma), and strata corresponding to collapse of internal edges (by Jacobi identity). The remaining strata correspond to collapsing of a vertex of the first type and a vertex of the second type. Figures below illustrate different cases of such boundary strata (for the first component): here $[x, B]\cdot\partial_x A(x,y) = \frac{d}{d\epsilon} A(x+\epsilon[x,B], y)|_{\epsilon=0}$ and $[y, B]\cdot\partial_y A(x,y) = \frac{d}{d\epsilon} A(x, y+\epsilon[y,B])|_{\epsilon=0}$.

- Fig. 6 computes terms of the type $[x, \omega_{\Rsh B} B(x,y)] \cdot \partial_x (\omega_{\Rsh A} A(x,y))$.
- Fig. 7 represents terms of the type $[y, \omega_{B\Lsh} B(x,y)] \cdot \partial_y (\omega_{\Rsh A} A(x,y))$.

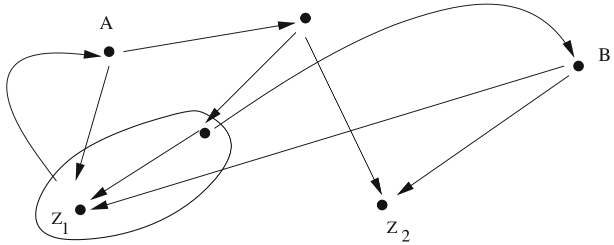

Fig. 6. $[x, \omega_{\Rsh B} B(x,y)] \cdot \partial_x (\omega_{\Rsh A} A(x,y))$

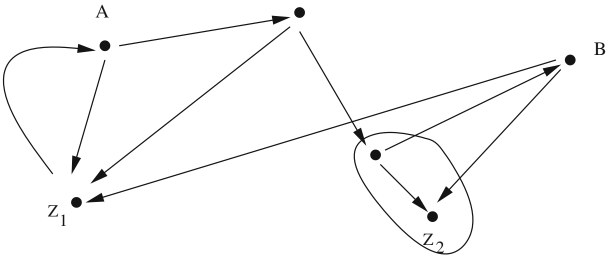

Fig. 7. $[y, \omega_{B\Lsh} B(x, y)] \cdot \partial_y(\omega_{\Rsh A} A(x, y))$

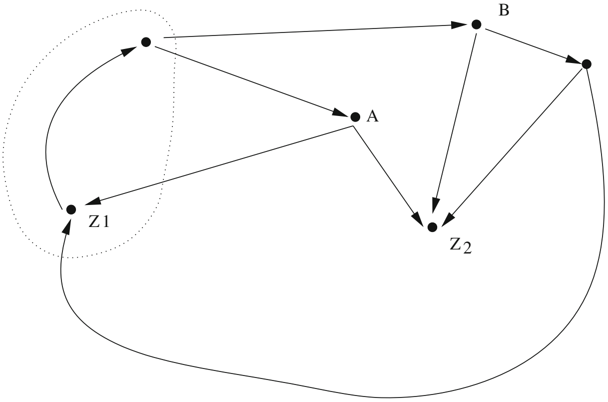

Fig. 8. $[\omega_{\Rsh A} A(x, y), \omega_{\Rsh B} B(x, y)]$

– Fig. 8 computes terms of the type $[\omega_{\Rsh A} A(x, y), \omega_{\Rsh B} B(x, y)]$. This term appears only once. It corresponds to the componentwise bracket in $\mathfrak{lie}_2 \times \mathfrak{lie}_2$,

$$\frac{1}{2}[\omega_2(x, y), \omega_2(x, y)]_{\mathfrak{lie}_2}.$$

These are exactly the three terms of the bracket $\frac{1}{2}[\omega_2, \omega_2]_{\mathfrak{tder}}$ (see Eq. (1)). By Eq. (12), one gets

$$\int_{\Delta_\xi} \left(\mathrm{d}\omega_2 + \frac{1}{2}[\omega_2, \omega_2] \right) = 0.$$

Since the curvature $\mathrm{d}\omega_2 + \frac{1}{2}[\omega_2, \omega_2]$ is a continuous function of ξ, we conclude that it vanishes on $C_{2,0}$. $\square$

3.2. Parallel transport and symmetries. In this section we discuss various properties of the connection ω_2, including the induced holonomies and their symmetries.

3.2.1. Parallel transport. Since ω_2 is flat, the equation

$$\mathrm{d}\, g = -g\omega_2,$$

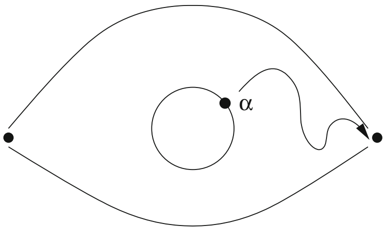

Fig. 9. A simple path from Duflo-Kontsevich star product to standard product

has a local solution on $C_{2,0}$ with values in $TAut_2 = \exp(\mathfrak{tder}_2)$. By abuse of notations we write $(u, v) \in TAut_2$ for an element acting on generators by $x \mapsto \mathrm{Ad}_u x = uxu^{-1}$, $y \mapsto \mathrm{Ad}_v y = vyv^{-1}$.

Take the initial data $g_\alpha = 1$ for α on the iris of $\overline{C}_{2,0}$ (see Fig. 9), and consider a path from α to ξ. The value at ξ for the parallel transport is well-defined since the connection is integrable, and by the flatness property it only depends on the homotopy class of the path. Integrating Eq. (8), we obtain $g_\xi(\mathrm{ch}_\xi(x, y)) = \mathrm{ch}_\alpha(x, y) = x + y$.

Recall [1] that for $\xi = (0, 1)$ this parallel transport F defines a solution of the Kashiwara-Vergne conjecture [9]. We conclude that this solution is independent of the choice of a path in the trivial homotopy class (the straight line joining $\alpha = 0$ and $\xi = (0, 1)$).

3.2.2. Holonomy. Solutions of equation $\mathrm{d}\,g = -g\omega_2$ are not globally defined on $\overline{C}_{2,0}$ because of the holonomy around the iris.

Lemma 1. *The restriction of* ω_2 *to the iris is equal to* $\omega_\theta = \frac{\mathrm{d}\theta}{2\pi}(y, x)$. *The holonomy around the Iris* $H_{2\pi}$ *is given by the inner automorphism* $(\exp(x+y), \exp(x+y)) \in TAut_2$.

Proof. When the point $\xi \in \overline{C}_{2,0}$ reaches the iris, the Kontsevich angle map degenerates to the Euclidean angle map on the complex plane, and the connection ω_2 is replaced by ω_2^∞,

$$\omega_2^\infty = \left(\sum w^\infty_{\Rsh A} A(x, y), \sum w^\infty_{B\Lsh} B(x, y)\right).$$

Since the Euclidean angle is rotation invariant, so is the 1-form $w^\infty_{\Rsh A}$. Hence, it is sufficient to compute $\int_{\mathbb{T}^1} w^\infty_{\Rsh A}$. By the Kontsevich Vanishing Lemma (see [11] § 6.6), integrals of 3 and more angle 1-forms vanish. Therefore, for A a nontrivial graph one gets $\int_{\mathbb{T}^1} w^\infty_{\Rsh A} = 0$ which implies $w^\infty_{\Rsh A} = 0$. As a result, we obtain the connection ω_θ by adding two trivial graph contributions,

$$\omega_\theta = \frac{\mathrm{d}\theta}{2\pi}(y, 0) + \frac{\mathrm{d}\theta}{2\pi}(0, x) = \frac{\mathrm{d}\theta}{2\pi}(y, x).$$

Let's integrate the equation $\mathrm{d}_\theta g = -g\omega_\theta$ over the boundary of the iris. Note that $t = (y, x)$ is actually an inner derivation since $t(x) = [x, y] = [x, x + y]$ and $t(y) = [y, x] = [y, x + y]$. We conclude that the parallel transport around the iris is given by $H_\theta = \exp(\theta t/2\pi) = (\exp(\theta(x + y)/2\pi), \exp(\theta(x + y)/2\pi))$. In particular, for $\theta = 2\pi$ we obtain $H_{2\pi} = (\exp(x + y), \exp(x + y))$, as required. $\square$

3.2.3. Symmetries of the connection. Consider the following involutions on $\overline{C}_{2,0}$:

$$\sigma_1 : (z_1, z_2) \mapsto (z_2, z_1) \quad \text{and} \quad \sigma_2 : (z_1, z_2) \mapsto (-\bar{z}_1, -\bar{z}_2).$$

Identifying $\overline{C}_{2,0}$ with the Kontsevich eye (see Fig. 1), σ_1 is the reflection with respect to the center of the eye, and σ_2 is the reflection with respect to the vertical axis (see Fig. 1).

We shall denote by τ_1 and τ_2 the following involutions of $\mathfrak{tder}_2$,

$$\begin{aligned}\tau_1 &: (F(x, y), G(x, y)) \mapsto (G(y, x), F(y, x)),\\ \tau_2 &: (F(x, y), G(x, y)) \mapsto (F(-x, -y), G(-x, -y)).\end{aligned}$$

They lift to involutions of $TAut_2$.

Proposition 1. *The connection ω_2 verifies*

$$\sigma_1^*(\omega_2) = \tau_1(\omega_2), \quad \sigma_2^*(\omega_2) = \tau_2(\omega_2).$$

Proof. The involution σ_1 simply exchanges the colors x and y of all graphs which induces the involution τ_1 on $\mathfrak{tder}_2$.

The involution σ_2 flips the sign of the one form $d\phi_e$ for each edge (since the reflection changes sign of the Euclidean angle), and changes the orientation of each integration over a complex variable. Hence, for a graph with n internal vertices we collect -1 to the power $(2n+1)+n \equiv n+1 \pmod 2$. Corresponding rooted trees have exactly $n+1$ leaves. Hence, one should change a sign of each leaf which results in applying the involution τ_2. □

Let $F \in TAut_2$ be the parallel transport of the equation $\mathrm{d}g = -g\omega_2$ for the straight path between the position 0 on the iris to the right corner of the eye $\overline{C}_{2,0}$. Since the path is invariant under the composition $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$, the parallel transport is invariant under $\tau = \tau_1\tau_2 = \tau_2\tau_1$, $\tau(F) = F$.

In order to discuss the involution τ_1 and τ_2 separately, we need the following lemma.

Lemma 2. *The parallel transport along the lower eyelid in the counter-clockwise direction is equal to $R = (\exp(y), 1) \in TAut_2$.*

Proof. The connection restricted to the lower eyelid has a trivial second component because edges starting from the real line give rise to a vanishing 1-form. Write the corresponding parallel transport as $R = (g(x, y), 1) \in TAut_2$. Integrating the equation $\mathrm{d}\ \mathrm{ch}_{\xi}(x, y) = \omega_2(\mathrm{ch}_{\xi}(x, y))$ along the lower eyelid, we obtain

$$R(\mathrm{ch}(x, y)) = \mathrm{ch}(\mathrm{Ad}_{g(x,y)}x, y) = \mathrm{ch}(y, x),$$

and this equation implies $g(x, y) = \exp(y)$, as required. □

Note that the path along the upper eyelid (oriented in the counter-clockwise direction) can be obtained by applying the involution σ_1 to the lower eyelid. Hence, the corresponding parallel transport is given by $\tau_1(R) = R^{2,1}$.

Proposition 2. *The element $F \in TAut_2$ verifies the following identities*:

$$F = e^{t/2}\tau_1(F)\tau_1(R^{-1}) = e^{-t/2}\tau_1(F)R.$$

Proof. These equations express the flatness condition for two contractible paths shown on Fig.10. □

These equations can be re-interpreted as the property of the parallel transport under the involution τ_1,

$$F^{2,1} = \tau_1(F) = e^{-t/2}FR^{2,1} = e^{t/2}FR^{-1}.$$

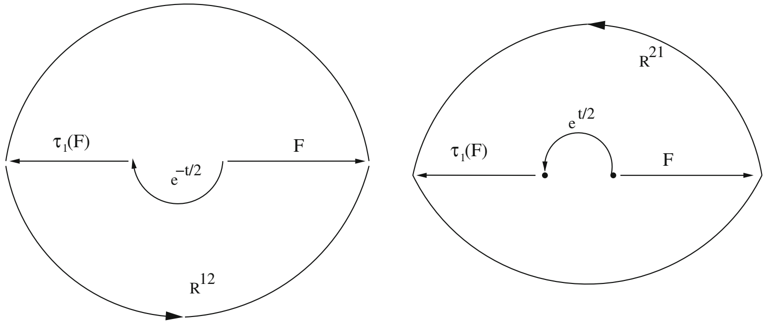

Fig. 10. The R matrix

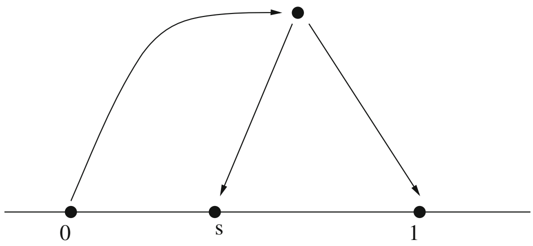

Fig. 11. A simple contribution in ω_3^∞

4. Connection ω_3 and Associators

4.1. Connection ω_3^∞. An important element of our construction is the connection ω_3^∞ which is built using the Kontsevich technique applied to the complex plane equipped with the Euclidean angle form.

Example 2. *Consider ω_3^∞ for 3 points situated on the real line at the positions* $0, s, 1$ (*see Fig.* 11). *The Euclidean angle form is* $\mathrm{d}\theta$ *with* $\tan(\theta) = \frac{y_2 - y_1}{x_2 - x_1}$. *One of the simplest trees is shown on Fig.* 11. *The corresponding 3-form is given by the following expression*:

$$\frac{1}{1+(\frac{y}{x})^2}\mathrm{d}\left(\frac{y}{x}\right) \wedge \frac{1}{1+(\frac{y}{x-s})^2}\mathrm{d}\left(\frac{y}{(x-s)}\right) \wedge \frac{1}{1+(\frac{y}{x-1})^2}\mathrm{d}\left(\frac{y}{(x-1)}\right)$$

$$= -\frac{y^2}{(x^2+y^2)((x-s)^2+y^2)((x-1)^2+y^2)}dx \wedge dy \wedge ds. \tag{13}$$

By [2] §I.1*, the orientation is given by $-dx \wedge dy \wedge ds$, and one gets*

$$\omega_{\Gamma^{(1)}}^\infty = -\frac{1}{8\pi^2}\left(\frac{\log(1-s)}{s} + \frac{\log(s)}{(1-s)}\right)\mathrm{d}s.$$

This 1-form is integrable (semi-algebraic), and one has $\int_0^1 \omega_{\Gamma^{(1)}}^\infty = \frac{1}{24}$.

Remark 3. Let $\alpha : z \mapsto \bar{z}$ be a the complex conjugation, and let κ be an involution of $\mathfrak{tder}_3$ defined by formula

$$(a(x,y,z), b(x,y,z), c(x,y,z)) \mapsto (a(-x,-y,-z), b(-x,-y,-z), c(-x,-y,-z)).$$

Then, $\alpha^*\omega_3^\infty = \kappa(\omega_3^\infty)$. The proof is similar to the one of Proposition 1.

Proposition 3. *Connection ω_3^∞ is flat and takes values in $\mathfrak{krv}_3$.*

Proof. The flatness condition $d\omega_3^\infty + \frac{1}{2}[\omega_3^\infty, \omega_3^\infty] = 0$ is obtained by replacing the hyperbolic angle form on the upper half-plane by the Euclidean angle form on the complex plane in the proof of Theorem 2.

Define

$$\mathrm{ch}_\xi^\infty(x, y, z) = x + y + z + \sum_{n \geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{Lie type } (n,3)}} w_\Gamma^\infty(\xi)\Gamma(x, y, z),$$

and

$$\mathrm{duf}_\xi^\infty(x, y, z) = \sum_{n \geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{wheel type } (\mathrm{n},3)}} \frac{w_\Gamma^\infty(\xi)}{m_\Gamma}\Gamma(x, y, z).$$

Similarly to (8) and (9), ω_3^∞ satisfies equations

$$\begin{aligned} d\,\mathrm{ch}_\xi^\infty(x, y, z) &= \omega_3^\infty(\mathrm{ch}_\xi^\infty(x, y, z)), \\ d\,\mathrm{duf}_\xi^\infty(x, y, z) &= \omega_3(\mathrm{duf}_\xi^\infty(x, y, z)) + \mathrm{div}(\omega_3). \end{aligned}$$

By the Kontsevich Vanishing Lemma (Lemma 6.6 in [11]), $w_\Gamma^\infty(\xi) = 0$ for all nontrivial graphs. Hence, $\mathrm{ch}_\xi^\infty(x, y, z) = x + y + z$ and $\mathrm{duf}_\xi^\infty(x, y, z) = 0$. Therefore, the differential equations for ω_3^∞ yield

$$\omega_3^\infty(x + y + z) = 0, \quad \mathrm{div}(\omega_3^\infty) = 0.$$

That is, ω_3^∞ takes values in $\mathfrak{krv}_3$ as required. □

Results of the previous proposition extend to connections ω_n^∞.

4.2. Associator. We will use the following notation. Recall that $T = (u, v) \in TAut_2$ is an automorphism of $\mathfrak{lie}_2$ acting by

$$(x, y) \mapsto (\mathrm{Ad}_u x, \mathrm{Ad}_v y).$$

We denote $T^{1,2} = (u(x, y), v(x, y), 1) \in TAut_3, T^{12,3} = (u(x+y, z), u(x+y, z), v(x+y, z)) \in TAut_3$, *etc*. For $F \in TAut_2$ the parallel transport from the iris to the right corner of the eye $\overline{C}_{2,0}$, we define

$$\Phi = F^{1,23}F^{23}(F^{12,3}F^{12})^{-1} \in TAut_3.$$

This element is the main topic of study in this section.

Proposition 4. *The element Φ coincides with the parallel transport for the equation $dg = -g\omega_3^\infty$ between positions* 1(23) *to* (12)3.

Proof. Consider the following path in the configuration space $\overline{C}_{3,0}$:

First, place z_1, z_2, z_3 on the stratum at infinity and move them along the horizontal line (the real axis of the complex plane at infinity) from the position 1(23) (z_2 and z_3 collapsed) to the position (12)3 (z_1 and z_2 collapsed). The connection at infinity is ω_3^∞, and we denote the corresponding parallel transport by Φ^∞.

Next, make z_3 descend from the stratum at infinity to plus infinity of the real axis of the upper half-plane (this corresponds to moving to the right corner of the eye for the points (12) and 3). On this stratum, the connection is $\omega_2^{12,3}$, and the parallel transport is given by $F^{12,3}$.

Continue with descending both z_1 and z_2 to the real axis. The connection on this stratum is $\omega_2^{1,2}$, and the parallel transport gives $F^{1,2}$.

Then, move z_2 to the vicinity of z_3 along the real axis of the upper half-plane. The parallel transport is trivial since the connection vanishes along the real axis.

Finally, lift z_2 and z_3 from the real axis and make them collapse on each other (parallel transport $(F^{2,3})^{-1}$), and lift z_1 from the real axis and make it collapse with $z_2 = z_3$ (parallel transport $(F^{1,23})^{-1}$).

Thus, we made a loop and returned to the position 1(23) at infinity. This loop is contractible, and the total parallel transport is trivial by the flatness property of the connection. Hence,

$$\Phi^\infty F^{12,3} F^{1,2} (F^{2,3})^{-1} (F^{1,23})^{-1} = 1,$$

and we obtain

$$\Phi^\infty = F^{1,23} F^{2,3} (F^{12,3} F^{1,2})^{-1} = \Phi,$$

as required. □

We have $\Phi = F^{1,23} F^{23} (F^{12,3} F^{12})^{-1}$, and the first term of Φ is given by

$$\Phi(x, y, z) = 1 - \frac{1}{24}([y, z], -[x, z], [y, z]) + \cdots = 1 + \frac{1}{24}[t^{1,2}, t^{2,3}] + \cdots ,$$

with $t^{1,2} = (y, x, 0)$ and $t^{2,3} = (0, z, y)$. Here we used the fact that $\int_0^1 \omega_{\Gamma^{(1)}}^\infty = \frac{1}{24}$ (see the example considered above), and the minus sign is coming from the orientation of the boundary stratum $C_3 \subset \partial C_{3,0}$. The main properties of the element Φ are summarized in the following theorem.

Theorem 3. *The element Φ satisfies associator axioms.*

Proof. The axioms to verify are as follows: Φ is a group like element and it is a solution of the following equations:

$$\Phi^{3,2,1}\Phi^{1,2,3} = 1, \tag{i}$$

$$\Phi^{1,2,34}\Phi^{12,3,4} = \Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3}, \tag{ii}$$

$$\begin{aligned}\exp\left(\pm\frac{1}{2}t_{12}\right)\Phi^{3,1,2}\exp\left(\pm\frac{1}{2}t_{13}\right)\Phi^{2,3,1}\exp\left(\pm\frac{1}{2}t_{23}\right)\Phi^{1,2,3}\\ = \exp\left(\pm\frac{1}{2}(t_{12}+t_{13}+t_{23})\right).\end{aligned} \tag{iii}$$

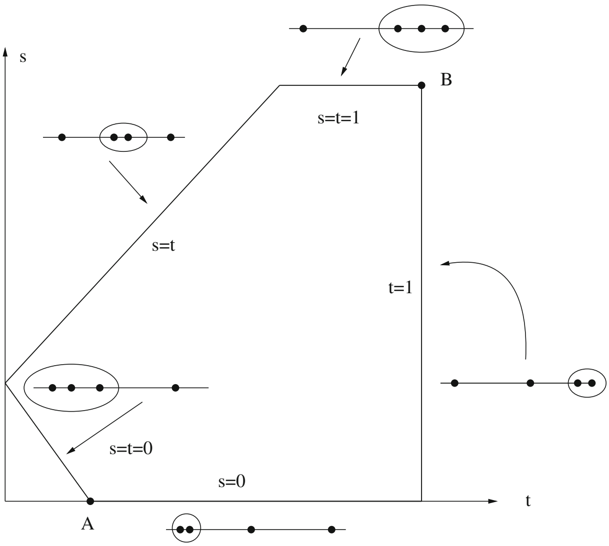

Fig. 12. The compactification space of 4 positions on a line

(i) - $(\Phi^{1,2,3})^{-1}$ is the parallel transport between positions 1(23) to (12)3. Let β be the reflection with respect to the vertical axis. Then, similar to Proposition 1, we obtain $\beta^*\omega_3^\infty = (\omega_3^\infty)^{3,2,1}$. Hence, we get $\Phi^{3,2,1} = (\Phi^{1,2,3})^{-1}$, as required.

(iii) - Consider the following path:

$$1(23) \mapsto (12)3 \mapsto (21)3 \mapsto 2(13) \mapsto 2(31) \mapsto (23)1 \mapsto 1(23).$$

Here the last step is by moving the collapsed pair (23) around the point 1 along the iris of the corresponding $\overline{C}_{2,0}$ stratum. By the flatness property, the total parallel transport is trivial. This gives exactly the pair of hexagonal equations (iii) by using the equation (i) and the fact that $c = t_{12} + t_{13} + t_{23}$ is central in $\mathfrak{sder}_3$ (see Proposition 3.4, [3]). The plus or minus sign in equation (iii) depends on the choice of the (clockwise or anti-clockwise) semi-circle for each exchange of two points (*e.g.* 1 moves above 2 or below 2 in the move $(12)3 \mapsto (21)3$).

(ii) - Consider four points $z_1 = 0, z_2 = s, z_3 = t, z_4 = 1$ on the horizontal line (the real axis of the complex plane) representing a point of the configuration space of the complex plane placed at infinity of $\overline{C}_{4,0}$. The path

$$((12)3)4 \mapsto (1(23))4 \mapsto 1((23)4) \mapsto 1(2(34))$$

is contractible. Hence, the parallel transport defined by the flat connection ω_4^∞ is trivial. It is easy to see that it reproduces the pentagon equation (ii) (see Fig. 12).

□

Note that Φ is an element of the group KRV_3 which contains the subgroup $T_3 = \exp(\mathfrak{t}_3)$. If Φ is actually an element of T_3, it becomes a Drinfeld associator. Since Φ is even ($\kappa(\Phi) = \Phi$, by Remark 3), it does not coincide with the Knizhnik-Zamolodchikov associator (the only known Drinfeld associator defined by an explicit formula).

Conjecture 1. *The element Φ is a Drinfeld associator. That is, $\Phi \in T_3 \subset KRV_3$.*

In a recent work [12], Severa and Willwacher prove this conjecture affirming that the element Φ is indeed a new Drinfeld associator admitting a presentation as a parallel transport of the flat connection ω_3^∞ defined by explicit formulas.

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References

1. Alekseev, A., Meinrenken, E.: On the Kashiwara-Vergne conjecture. Invent. Math. **164**(3), 615–634 (2006)
2. Arnal, D., Manchon, D., Masmoudi, M.: Choix des signes pour la formalité de Kontsevich. Pacific J. Math. **203**, 23–66 (2002)
3. Alekseev, A., Torossian, C.: *The Kashiwara-Vergne conjecture and Drinfeld's associators*. http://arxiv.org/abs/0802.4300v1[math.QA], 2008
4. Andler, M., Sahi, S., Torossian, C.: Convolution of invariant distributions: proof of the Kashiwara-Vergne conjecture. Lett. Math. Phys. **69**, 177–203 (2004)
5. Cattaneo, A.S., Keller, B., Torossian, C., Bruguières, A.: *Déformation, quantification, théorie de Lie*. Collection Panoramas et Synthèse no. **20**, SMF, 2005.
6. Dito, G.: Kontsevich star product on the dual of a Lie algebra. Lett. Math. Phys. **48**(4), 307–322 (1999)
7. Drinfeld, V.G.: *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* Gal$(\overline{Q}/Q)$. (Russian) Algebra i Analiz 2, no. 4, 149–181 (1990); translation in Leningrad Math. J. 2, no. 4, 829–860 (1991)
8. Felder, G., Willwacher, T.: *On the (ir)rationality of Kontsevich weights*. http://arxiv.org/abs/0808.2762v2[math.QA], 2008
9. Kashiwara, M., Vergne, M.: The Campbell-Hausdorff formula and invariant hyperfunctions. Inventiones Math. **47**, 249–272 (1978)
10. Kathotia, V.: Kontsevich's universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula. Internat. J. Math. **11**(4), 523–551 (2000)
11. Kontsevich, M.: Deformation quantization of Poisson manifolds, I. Lett. Math.Phys. **66**(3), 157–216 (2003)
12. Severa, P., Willwacher, T.: *Equivalence of formalities of the little discs operad*. http://arxiv.org/abs/0905.1789v1[math.QA], 2009
13. Shoikhet, B.: Vanishing of the Kontsevich integrals of the wheels. EuroConférence Moshe Flato 2000, Part II (Dijon). Lett. Math. Phys. **56**(2), 141–149 (2001)
14. Torossian, C.: Sur la conjecture combinatoire de Kashiwara-Vergne. J. Lie Theory **12**(2), 597–616 (2002)

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